Technical Appendix

Not to be published

A Price index and individual demand

The aggregate consumption in the home country is composed by home finished goods and foreign finished goods:

$$C = 2C_h^{\frac{1}{2}} C_f^{\frac{1}{2}} \tag{A.1}$$

Thus, the consumption-based price index P, which is the minimum nominal expenditure to purchase one unit of aggregate consumption, can be found by solving the following minimization problem:

$$\min_{(C_h, C_f)} \quad Z = P_h C_h + P_f C_f \tag{A.2}$$

$$\text{S.t} \quad 2C_h^{\frac{1}{2}} C_f^{\frac{1}{2}} = 1 \tag{A.3}$$

So the CPI price index is given by:

$$P = P_h^{\frac{1}{2}} P_f^{\frac{1}{2}} \tag{A.4}$$

Takeing the prices for the home finished goods and foreign finished goods as given, the consumer allocates a given level of aggregate consumption among the home and foreign sub-aggregate finished goods:

$$\max_{\{C_h, C_f\}} \quad C = 2C_h^{\frac{1}{2}} C_f^{\frac{1}{2}} \tag{A.5}$$

$$s.t \quad P_h C_h + P_f C_f = PC \tag{A.6}$$

Thus, we have

$$C_h = \frac{1}{2}\frac{PC}{P_h}, \qquad C_f = \frac{1}{2}\frac{PC}{P_f} \tag{A.7}$$

Similarly, given the definition of the sub-aggregate consumption of the home and foreign finished goods $C_h = [\int_0^1 C_h(i)^{\frac{\lambda-1}{\lambda}} di]^{\frac{\lambda}{\lambda-1}}$ and $C_f = [\int_0^1 C_f(i)^{\frac{\lambda-1}{\lambda}} di]^{\frac{\lambda}{\lambda-1}}$ the price indexes for sub-aggregate home and foreign finished goods are given by :

$$P_h = [\int_0^1 P_h(i)^{1-\lambda} di]^{\frac{1}{1-\lambda}}, \qquad P_f = [\int_0^1 P_f(i)^{1-\lambda} di]^{\frac{1}{1-\lambda}} \tag{A.8}$$

Then the optimal demand for individual finished goods can be derived as:

$$C_h(i) = [\frac{P_h(i))}{P_h}]^{-\lambda} C_h, \qquad C_f(i) = [\frac{P_f(i))}{P_f}]^{-\lambda} C_f \tag{A.9}$$

The price index and individual demand for intermediate goods can be derived analogously.

B Expected welfare

Solving for Ec First, substituting the optimal pricing schedules $\tilde{P}_h$ and $\tilde{P}_f$ into the definition of the price indices P_h and P_f^*, we have [25]

$$P_h = (\hat{\lambda}\hat{\phi})\frac{[E(\frac{S^{\frac{1}{2}}}{\theta_F}C^{1-\rho})][E(\frac{W}{\theta_I}\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F})]^{\frac{1}{2}}[E(\frac{W^*}{\theta_I^*}\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F})]^{\frac{1}{2}}}{[EC^{1-\rho}][E(\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F})]^{\frac{1}{2}}[E(\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F})]^{\frac{1}{2}}} \tag{B.1}$$

$$P_f^* = (\hat{\lambda}\hat{\phi})\frac{[E(\frac{S^{-\frac{1}{2}}}{\theta_F^*}C^{1-\rho})][E(\frac{W}{\theta_I}\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*})]^{\frac{1}{2}}[E(\frac{W^*}{\theta_I^*}\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*})]^{\frac{1}{2}}}{[EC^{1-\rho}][E(\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*})]^{\frac{1}{2}}[E(\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*})]^{\frac{1}{2}}} \tag{B.2}$$

Putting the above two equations together, we have

$$P_hP_f^* = (\hat{\lambda}\hat{\phi})^2\frac{[E(\frac{W}{\theta_I}\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F})]^{\frac{1}{2}}[E(\frac{W^*}{\theta_I^*}\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F})]^{\frac{1}{2}}[E(\frac{W}{\theta_I}\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*})]^{\frac{1}{2}}[E(\frac{W^*}{\theta_I^*}\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*})]^{\frac{1}{2}}}{[EC^{1-\rho}]^2} \tag{B.3}$$

Using the labor supply function to eliminate the predetermined prices on both sides, we have

$$1 = (\hat{\lambda}\hat{\phi}\eta)^2\frac{[E(\frac{SC}{\theta_F\theta_I})]^{\frac{1}{2}}[E(\frac{C}{\theta_F\theta_I^*})]^{\frac{1}{2}}[E(\frac{C}{\theta_F^*\theta_I})]^{\frac{1}{2}}[E(\frac{S^{-1}C}{\theta_F^*\theta_I^*})]^{\frac{1}{2}}}{[EC^{1-\rho}]^2} \tag{B.4}$$

Using the properties of log-normal distribution, taking log of (B.4), we have

$$\begin{aligned} 0 \;=\; & \log[(\hat{\lambda}\hat{\phi}\eta)^2] + \frac{1}{2}[Ec + \frac{1}{2}\sigma_s^2 + \frac{1}{2}\sigma_u^2 + \frac{1}{2}\sigma_v^2 + \frac{1}{2}\sigma_c^2 - \sigma_{su} - \sigma_{sv} - \sigma_{cu} - \sigma_{cv} + \sigma_{cs} + \sigma_{uv}] \\ & + \frac{1}{2}[Ec + \frac{1}{2}\sigma_u^2 + \frac{1}{2}\sigma_{v^*}^2 + \frac{1}{2}\sigma_c^2 - \sigma_{cu} - \sigma_{cv^*}] \\ & + \frac{1}{2}[Ec + \frac{1}{2}\sigma_{u^*}^2 + \frac{1}{2}\sigma_v^2 + \frac{1}{2}\sigma_c^2 - \sigma_{cu^*} - \sigma_{cv}] \\ & + \frac{1}{2}[Ec + \frac{1}{2}\sigma_s^2 + \frac{1}{2}\sigma_{u^*}^2 + \frac{1}{2}\sigma_{v^*}^2 + \frac{1}{2}\sigma_c^2 + \sigma_{su^*} + \sigma_{sv^*} - \sigma_{cu^*} - \sigma_{cv^*} - \sigma_{cs} + \sigma_{u^*v^*}] \\ & - 2[(1-\rho)Ec + \frac{(1-\rho)^2}{2}\sigma_c^2] \end{aligned} \tag{B.5}$$

The next step is to derive the mean of the log consumption in terms of variance and covariance of the log consumption, the log exchange rate, and the productivity shocks.

$$\begin{aligned} Ec \;=\; & \frac{-ln[\hat{\lambda}\hat{\phi}\eta]}{\rho} - \frac{(2-\rho)}{2}\sigma_c^2 - \frac{1}{4\rho}\sigma_s^2 - \frac{1}{4\rho}[\sigma_u^2 + \sigma_{u^*}^2 + \sigma_v^2 + \sigma_{v^*}^2 + \sigma_{uv} + \sigma_{u^*v^*}] \\ & + \frac{1}{4\rho}[\sigma_{su} - \sigma_{su^*}] + \frac{1}{4\rho}[\sigma_{sv} - \sigma_{sv^*}] + \frac{1}{2\rho}[\sigma_{cu} + \sigma_{cu^*} + \sigma_{cv} + \sigma_{cv^*}] \end{aligned} \tag{B.6}$$

Solving for EL The intermediate goods market clearing condition implies

$$\theta_I L = X_h + X_h^* = \frac{1}{2}\frac{\Lambda}{\tilde{P}_h}\frac{PC}{P_h} + \frac{1}{2}\frac{\Lambda^*}{\frac{\tilde{P}_{hf}}{S}}\frac{P^*C^*}{P_f^*} \tag{B.7}$$

[25] For simplicity, we have used the fact that $C = C^*$ here.

Substituting $\tilde{P_h}$,$\tilde{P_{hf}}$, P_h and P_f into the above equation, we have

$$
\begin{aligned}
L &= \frac{1}{2\hat{\lambda}\hat{\phi}} \frac{PC}{\theta_I} \frac{\tilde{P_h}^{\frac{1}{2}}(S\tilde{P_{fh}^*})^{\frac{1}{2}}}{\theta_F} \frac{E(\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F})}{E[\frac{W}{\theta_I}\frac{S^{\frac{1}{2}}C^{1-\rho}}{\theta_F}]} \frac{E(C^{1-\rho})}{E[\frac{\tilde{P_h}^{\frac{1}{2}}(S\tilde{P_{fh}^*})^{\frac{1}{2}}}{\theta_F}C^{1-\rho}]} \\
&+ \frac{1}{2\hat{\lambda}\hat{\phi}} \frac{P^*C}{\theta_I} \frac{(\frac{\tilde{P_{hf}}}{S})^{\frac{1}{2}}(\tilde{P_f^*})^{\frac{1}{2}}S}{\theta_F^*} \frac{E(\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*})}{E[\frac{W}{\theta_I}\frac{S^{-\frac{1}{2}}C^{1-\rho}}{\theta_F^*}]} \frac{E(C^{1-\rho})}{E[\frac{(\frac{\tilde{P_{hf}}}{S})^{\frac{1}{2}}(\tilde{P_f^*})^{\frac{1}{2}}}{\theta_F^*}C^{1-\rho}]}
\end{aligned} \tag{B.8}
$$

Using the labor supply function to eliminate the predetermined terms, and then taking expectation, we have

$$EL = \frac{1}{\hat{\lambda}\hat{\phi}\eta} EC^{1-\rho} \tag{B.9}$$

Expected welfare We may rewrite the expected utility of the home country as

$$EU = \frac{\lambda\phi - (1-\rho)(\lambda-1)(\phi-1)}{(1-\rho)\lambda\phi} EC^{1-\rho} \tag{B.10}$$

Since $EC^{1-\rho} = \exp{(1-\rho)}(Ec + \frac{1-\rho}{2}\sigma_c^2)$, the monetary authority's problem is equivalent to maximize $U_0 = Ec + \frac{1-\rho}{2}\sigma_c^2$,

$$
\begin{aligned}
U_0 &= \frac{-ln[\hat{\lambda}\hat{\phi}\eta]}{\rho} - \frac{1}{2}\sigma_c^2 - \frac{1}{4\rho}\sigma_s^2 - \frac{1}{4\rho}[\sigma_u^2 + \sigma_{u^*}^2 + \sigma_v^2 + \sigma_{v^*}^2 + \sigma_{uv} + \sigma_{u^*v^*}] \\
&+ \frac{1}{4\rho}[\sigma_{su} - \sigma_{su^*}] + \frac{1}{4\rho}[\sigma_{sv} - \sigma_{sv^*}] + \frac{1}{2\rho}[\sigma_{cu} + \sigma_{cu^*} + \sigma_{cv} + \sigma_{cv^*}]
\end{aligned} \tag{B.11}
$$

C Optimal money rules

Variance and covariance terms Given the money rules and the solution for the exchange rate and the consumption (4.1)-(4.4), we have

$$
\begin{aligned}
\sigma_c^2 = \ & \frac{1}{4\rho^2}[(a_1+b_1)^2\sigma_u^2 + (a_2+b_2)^2\sigma_{u^*}^2 + (a_3+b_3)^2\sigma_v^2 + (a_4+b_4)^2\sigma_{v^*}^2 \\
& +2(a_1+b_1)(a_3+b_3)\sigma_{uv} + 2(a_2+b_2)(a_4+b_4)\sigma_{u^*v^*}]
\end{aligned} \tag{C.1}
$$

$$
\begin{aligned}
\sigma_s^2 = \ & [(a_1-b_1)^2\sigma_u^2 + (a_2-b_2)^2\sigma_{u^*}^2 + (a_3-b_3)^2\sigma_v^2 + (a_4-b_4)^2\sigma_{v^*}^2 \\
& +2(a_1-b_1)(a_3-b_3)\sigma_{uv} + 2(a_2-b_2)(a_4-b_4)\sigma_{u^*v^*}]
\end{aligned} \tag{C.2}
$$

$$\sigma_{su} = (a_1-b_1)\sigma_u^2 + (a_3-b_3)\sigma_{uv}, \qquad \sigma_{su^*} = (a_2-b_2)\sigma_{u^*}^2 + (a_4-b_4)\sigma_{u^*v^*} \tag{C.3}$$

$$\sigma_{sv} = (a_3-b_3)\sigma_v^2 + (a_1-b_1)\sigma_{uv}, \qquad \sigma_{sv^*} = (a_4-b_4)\sigma_{v^*}^2 + (a_2-b_2)\sigma_{u^*v^*} \tag{C.4}$$

$$\sigma_{cu} = \frac{1}{2\rho}(a_1+b_1)\sigma_u^2 + \frac{1}{2\rho}(a_3+b_3)\sigma_{uv}, \qquad \sigma_{cu^*} = \frac{1}{2\rho}(a_2+b_2)\sigma_{u^*}^2 + \frac{1}{2\rho}(a_4+b_4)\sigma_{u^*v^*} \tag{C.5}$$

$$\sigma_{cv} = \frac{1}{2\rho}(a_3+b_3)\sigma_v^2 + \frac{1}{2\rho}(a_1+b_1)\sigma_{uv}, \qquad \sigma_{cv^*} = \frac{1}{2\rho}(a_4+b_4)\sigma_{v^*}^2 + \frac{1}{2\rho}(a_2+b_2)\sigma_{u^*v^*} \tag{C.6}$$

Substituting the above expressions into equation (B.11), we may express the objective functions of monetary authorities as functions of policy parameters.

Solution to the Nash game Now we could rewrite U_0 as

$$\begin{aligned}U_0 &= \frac{-ln[\hat{\lambda}\hat{\phi}\eta]}{\rho} - \frac{1}{8\rho^2}[(a_1+b_1)^2\sigma_u^2 + (a_2+b_2)^2\sigma_{u^*}^2 + (a_3+b_3)^2\sigma_v^2 + (a_4+b_4)^2\sigma_{v^*}^2 \\ &+2(a_1+b_1)(a_3+b_3)\sigma_{uv} + 2(a_2+b_2)(a_4+b_4)\sigma_{u^*v^*}] - \frac{1}{4\rho}[(a_1-b_1)^2\sigma_u^2 + (a_2-b_2)^2\sigma_{u^*}^2 \\ &+(a_3-b_3)^2\sigma_v^2 + (a_4-b_4)^2\sigma_{v^*}^2 + 2(a_1-b_1)(a_3-b_3)\sigma_{uv} + 2(a_2-b_2)(a_4-b_4)\sigma_{u^*v^*}] \\ &-\frac{1}{4\rho}[\sigma_u^2 + \sigma_{u^*}^2 + \sigma_v^2 + \sigma_{v^*}^2 + \sigma_{uv} + \sigma_{u^*v^*}] + \frac{1}{4\rho}[(a_1-b_1)\sigma_u^2 + (a_3-b_3)\sigma_{uv} - (a_2-b_2)\sigma_{u^*}^2 \\ &-(a_4-b_4)\sigma_{u^*v^*}] + \frac{1}{4\rho}[(a_3-b_3)\sigma_v^2 + (a_1-b_1)\sigma_{uv} - (a_4-b_4)\sigma_{v^*}^2 - (a_2-b_2)\sigma_{u^*v^*}] \\ &+\frac{1}{4\rho^2}[(a_1+b_1)\sigma_u^2 + (a_3+b_3)\sigma_{uv} + (a_2+b_2)\sigma_{u^*}^2 + (a_4+b_4)\sigma_{u^*v^*} + (a_3+b_3)\sigma_v^2 \\ &(a_1+b_1)\sigma_{uv} + (a_4+b_4)\sigma_{v^*}^2 + (a_2+b_2)\sigma_{u^*v^*}] \qquad \text{(C.7)}\end{aligned}$$

Since the objective functions of the home and the foreign monetary authorities are identical, the reaction functions are given by:

$$\frac{\partial U_0(a, b^N)}{a_i} = 0, \qquad \forall i = 1, 2, 3, 4 \qquad \text{(C.8)}$$

$$\frac{\partial U_0(a^N, b)}{b_i} = 0, \qquad \forall i = 1, 2, 3, 4 \qquad \text{(C.9)}$$

Specifically:

$$-\frac{1}{4\rho^2}[(a_1+b_1)\sigma_u^2+(a_3+b_3)\sigma_{uv}]-\frac{1}{2\rho}[(a_1-b_1)\sigma_u^2+(a_3-b_3)\sigma_{uv}]+\frac{1}{4\rho}[\sigma_u^2+\sigma_{uv}]+\frac{1}{4\rho^2}[\sigma_u^2+\sigma_{uv}] = 0$$

$$-\frac{1}{4\rho^2}[(a_2+b_2)\sigma_{u^*}^2+(a_4+b_4)\sigma_{u^*v^*}]-\frac{1}{2\rho}[(a_2-b_2)\sigma_{u^*}^2+(a_4-b_4)\sigma_{u^*v^*}]-\frac{1}{4\rho}[\sigma_{u^*}^2+\sigma_{u^*v^*}]+\frac{1}{4\rho^2}[\sigma_{u^*}^2+\sigma_{u^*v^*}] = 0$$

$$-\frac{1}{4\rho^2}[(a_3+b_3)\sigma_v^2+(a_1+b_1)\sigma_{uv}]-\frac{1}{2\rho}[(a_3-b_3)\sigma_v^2+(a_1-b_1)\sigma_{uv}]+\frac{1}{4\rho}[\sigma_{uv}+\sigma_v^2]+\frac{1}{4\rho^2}[\sigma_{uv}+\sigma_v^2] = 0$$

$$-\frac{1}{4\rho^2}[(a_4+b_4)\sigma_{v^*}^2+(a_2+b_2)\sigma_{u^*v^*}]-\frac{1}{2\rho}[(a_4-b_4)\sigma_{v^*}^2+(a_2-b_2)\sigma_{u^*v^*}]-\frac{1}{4\rho}[\sigma_{v^*}^2+\sigma_{u^*v^*}]+\frac{1}{4\rho^2}[\sigma_{v^*}^2+\sigma_{u^*v^*}] = 0$$

$$-\frac{1}{4\rho^2}[(a_1+b_1)\sigma_u^2+(a_3+b_3)\sigma_{uv}]+\frac{1}{2\rho}[(a_1-b_1)\sigma_u^2+(a_3-b_3)\sigma_{uv}]-\frac{1}{4\rho}[\sigma_u^2+\sigma_{uv}]+\frac{1}{4\rho^2}[\sigma_u^2+\sigma_{uv}] = 0$$

$$-\frac{1}{4\rho^2}[(a_2+b_2)\sigma_{u^*}^2+(a_4+b_4)\sigma_{u^*v^*}]+\frac{1}{2\rho}[(a_2-b_2)\sigma_{u^*}^2+(a_4-b_4)\sigma_{u^*v^*}]+\frac{1}{4\rho}[\sigma_{u^*}^2+\sigma_{u^*v^*}]+\frac{1}{4\rho^2}[\sigma_{u^*}^2+\sigma_{u^*v^*}] = 0$$

$$-\frac{1}{4\rho^2}[(a_3+b_3)\sigma_v^2+(a_1+b_1)\sigma_{uv}]+\frac{1}{2\rho}[(a_3-b_3)\sigma_v^2+(a_1-b_1)\sigma_{uv}]-\frac{1}{4\rho}[\sigma_{uv}+\sigma_v^2]+\frac{1}{4\rho^2}[\sigma_{uv}+\sigma_v^2] = 0$$

$$-\frac{1}{4\rho^2}[(a_4+b_4)\sigma_{v^*}^2+(a_2+b_2)\sigma_{u^*v^*}]+\frac{1}{2\rho}[(a_4-b_4)\sigma_{v^*}^2+(a_2-b_2)\sigma_{u^*v^*}]+\frac{1}{4\rho}[\sigma_{v^*}^2+\sigma_{u^*v^*}]+\frac{1}{4\rho^2}[\sigma_{v^*}^2+\sigma_{u^*v^*}] = 0$$

Thus, from these 8 reaction functions, we may derive the solution to the international monetary game. Substituting these optimal policy parameters into equation (B.11), we may get the maximized expected utility level under the sticky price equilibrium.

$$U_0(sticky) = \frac{-ln[\hat{\lambda}\hat{\phi}\eta]}{\rho} + \frac{1-\rho}{8\rho^2}[\sigma_u^2 + \sigma_{u^*}^2 + \sigma_v^2 + \sigma_{v^*}^2 + 2\sigma_{uv} + 2\sigma_{u^*v^*}] - \frac{1}{16\rho}[\sigma_u^2 + \sigma_{u^*}^2 + \sigma_v^2 + \sigma_{v^*}^2 - 2\sigma_{uv} - 2\sigma_{u^*v^*}] \tag{C.10}$$

From equation (C.10) and (3.8), equation (4.8) is obvious.